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DESIDERATA AND SUGGESTIONS.

By Professor Cayley, *Cambridge, England.*

No. 1.—THE THEORY OF GROUPS.

Substitutions, and (in connexion therewith) groups, have been a good deal studied; but only a little has been done towards the solution of the general problem of groups. I give the theory so far as is necessary for the purpose of pointing out what appears to me to be wanting.

Let α, β, . . . be functional symbols, each operating upon one and the same number of letters and producing as its result the same number of functions of these letters; for instance, $\alpha(x, y, z) = (X, Y, Z)$, where the capitals denote each of them a given function of (x, y, z).

Such symbols are susceptible of repetition and of combination; $\alpha^2(x, y, z) = \alpha(X, Y, Z)$, or $\beta\alpha(x, y, z) = \beta(X, Y, Z)$, $=$ in each case three given functions of (x, y, z), and similarly α^3, $\alpha^2\beta$, &c.

The symbols are not in general commutative, $\alpha\beta$ not $= \beta\alpha$; but they are associative, $\alpha\beta.\gamma = \alpha.\beta\gamma$, each $= \alpha\beta\gamma$, which has thus a determinate signification.

[The associativeness of such symbols arises from the circumstance that the definitions of α, β, γ, . . determine the meanings of $\alpha\beta$, $\alpha\gamma$, &c.: if α, β, γ . . were quasi-quantitative symbols such as the quaternion imaginaries i, j, k, then $\alpha\beta$ and $\beta\gamma$ might have by definition values δ and ε such that $\alpha\beta.\gamma$ and $\alpha.\beta\gamma$ ($= \delta\gamma$ and $\alpha\varepsilon$ respectively) have unequal values].

Unity as a functional symbol denotes that the letters are unaltered, $1(x, y, z) = (x, y, z)$; whence $1\alpha = \alpha 1 = \alpha$.

The functional symbols *may* be substitutions; $\alpha(x, y, z) = (y, z, x)$, the same letters in a different order: substitutions can be represented by the notation $\alpha = \dfrac{yzx}{xyz}$, the substitution which changes xyz into yzx, or as products of cyclical substitutions, $\alpha = \dfrac{yzx\,wu}{xyz\,uw}$, $= (xyz)(uw)$, the product of the cyclical interchanges x into y, y into z, and z into x; and u into w, w into u.

A set of symbols α, β, γ . . . such that the product $\alpha\beta$ of each two of them (in each order, $\alpha\beta$ or $\beta\alpha$,) is a symbol of the set, is a group. It is easily seen that 1 is a symbol of every group, and we may therefore give the definition in the form that a set of symbols, 1, α, β, γ . . satisfying the foregoing condition is a group. When the number of the symbols (or terms) is $= n$, then the group is of the nth order; and each symbol α is such that $\alpha^n = 1$, so that a group of the order n is, in fact, a group of symbolical nth roots of unity.

A group is defined by means of the laws of combination of its symbols: for the statement of these we may either (by the introduction of powers and products) diminish as much as may be the number of independent functional symbols, or else, using distinct letters for the several terms of the group, employ a square diagram as presently mentioned.

Thus in the first mode, a group is 1, β, β^2, α, $\alpha\beta$, $\alpha\beta^2$ ($\alpha^2 = 1$, $\beta^3 = 1$, $\alpha\beta = \beta^2\alpha$); where observe that these conditions imply also $\alpha\beta^2 = \beta\alpha$:

Or in the second mode calling the same group (1, α, β, γ, δ, ε), the laws of combination are given by the square diagram

	1	α	β	γ	δ	ε
1	1	α	β	γ	δ	ε
α	α	1	γ	β	ε	δ
β	β	ε	δ	α	1	γ
γ	γ	δ	ε	1	α	β
δ	δ	γ	1	ε	β	α
ε	ε	β	α	δ	γ	1

for the symbols (1, α, β, γ, δ, ε) are in fact $=$ (1, α, β, $\alpha\beta$, β^2, $\alpha\beta^2$).

The general problem is to find all the groups of a given order n; thus if $n = 2$, the only group is 1, α ($\alpha^2 = 1$); $n = 3$, the only group is 1, α, α^2 ($\alpha^3 = 1$); $n = 4$, the groups are 1, α, α^2, α^3 ($\alpha^4 = 1$), and 1, α, β, $\alpha\beta$ ($\alpha^2 = 1$, $\beta^2 = 1$, $\alpha\beta = \beta\alpha$);* $n = 6$, there are three groups, a group 1, α, α^2, α^3, α^4, α^5

* If $n = 5$, the only group is 1, a, a^2, a^3, a^4 ($a^5 = 1$). W. E. S.

($\alpha^6 = 1$) ; and two groups $1, \beta, \beta^2, \alpha, \alpha\beta, \alpha\beta^2$ ($\alpha^2 = 1.\ \beta^3 = 1$), viz: in the first of these $\alpha\beta = \beta\alpha$; while in the other of them (that mentioned above) we have $\alpha\beta = \beta^2\alpha,\ \alpha\beta^2 = \beta\alpha$.

But although the theory as above stated is a general one, including as a particular case the theory of substitutions, yet the general problem of finding all the groups of a given order n, is really identical with the apparently less general problem of finding all the groups of the same order n, which can be formed with the substitutions upon n letters; in fact, referring to the diagram, it appears that $1, \alpha, \beta, \gamma, \delta, \varepsilon$ may be regarded as substitutions performed upon the six letters $1, \alpha, \beta, \gamma, \delta, \varepsilon$, viz: 1 is the substitution unity which leaves the order unaltered, α the substitution which changes $1\alpha\beta\gamma\delta\varepsilon$ into $\alpha 1\gamma\beta\varepsilon\delta$, and so for $\beta, \gamma, \delta, \varepsilon$. This, however, does not in any wise show that the best or easiest mode of treating the general problem is thus to regard it as a problem of substitutions: and it seems clear that the better course is to consider the general problem in itself, and to deduce from it the theory of groups of substitutions.

CAMBRIDGE, *26th November, 1877.*